ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Final Examination

MATH-337
COMBINATORIAL NUMBER THEORY
SPRING SEMESTER 2023

07 JULY, 2023

First name: ____________________

Last name: ____________________

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Instructions

- This examination consists of 9 questions, out of which you have to answer 6. You can choose yourself which 6.
- Do not submit solutions for more than 6 questions. If you submit solutions to more, then only the ones with the least score will count towards your final grade.
- You have three hours (180 minutes) to complete this exam.
- The use of books, notes, calculators, computers, tablets or phones is prohibited.
- Write legibly and show all of your work. Unsupported answers may not earn credit. Cross out any work that you do not wish to have scored.
- Write your solutions only on the provided exam paper; do not use your own paper.
- You are permitted to use results from the course without proving them if you state and apply them correctly.
- Each question on this exam is graded out of 8 points using the same rubric as we did for homework assignments: mathematical correctness (worth 5 points) and proof-writing quality (worth 3 points). The maximal amount of points that you can score on this exam is 48.

Grading Table

(for examiner use only)

Question	Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
Total:	48	

1. (8 points) State and prove the Erdős-Szekeres' Theorem on points in convex position. If you use any results from the course then make sure to state and apply them correctly.

2. (8 points) Write down the definition of the topology on $\beta\mathbb{N}$ and then prove that this topology is compact Hausdorff.

3. (8 points) State and prove the Ellis-Numakura Lemma.

4. (8 points) Let $N \in \mathbb{N}$, $a, b, c \in \mathbb{Z}\setminus\{0\}$ and assume N is coprime to abc. Provide a proof that for all $A_1, A_2, A_3 \subseteq \{0, 1, \ldots, N-1\}$ one has

$$\left|\{(x, y, z) \in A_1 \times A_2 \times A_3 : ax + by + cz \equiv 0 \bmod N\}\right| = N^2 \sum_{\xi \in \mathbb{Z}_N} \hat{A}_1(a\xi)\hat{A}_2(b\xi)\hat{A}_3(c\xi),$$

where $\hat{A}_i : \mathbb{Z}_N \to \mathbb{C}$ denotes the Fourier transform of the indicator function of A_i.

5. (8 points) Let $A \subseteq \mathbb{N}$ be piecewise syndetic. Show that some shift of A contains a solution to the equation $x + y = z$, i.e., there are $t \in \mathbb{N} \cup \{0\}$ and $x, y, z \in A - t$ with $x + y = z$.

6. (8 points) Suppose all rational numbers in the interval $(0, 1)$ are colored using finitely many colors. Prove that there exists an infinite set of prime numbers $P \subseteq \mathbb{P}$ such that all numbers of the from $\frac{p}{q}$ for $p, q \in P$ and $p < q$ are monochromatic.

7. (8 points) Suppose $T \subseteq \mathbb{N}$ is thick. Prove that there exists an ultrafilter $p \in \beta\mathbb{N}$ such that $T - p = \mathbb{N}$.

8. (8 points) Let N be an even positive integer and $\varepsilon > 0$. Show that if $A \subseteq \{0, 1, \ldots, N-1\}$ is ε-pseudorandom then the set $B = \{n \in A : n \text{ is even}\}$ satisfies $\left|\frac{|A|}{N} - \frac{2|B|}{N}\right| \leqslant \varepsilon$.

9. (8 points) Prove that for all $\alpha \in (0, 1)$ one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(n\alpha)\log(n) = 0.$$